Stochastic Present Value & Jensen's Inequality

by

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Present Value

Present value of earnings for year t is the amount that would have to be invested today with an annual rate of return d, also called the discount rate, so as to compound into becoming $Earnings_t$ after t years:

$$Present\ Value\ of\ Earnings_t \cdot (1+d)^t = Earnings_t$$

$$\Leftrightarrow Present\ Value\ of\ Earnings_t = \frac{Earnings_t}{(1+d)^t}$$

The present value for n years is the sum:

$$Present\ Value = \sum_{t=1}^{n} Present\ Value\ of\ Earnings_t = \sum_{t=1}^{n} \frac{Earnings_t}{(1+d)^t}$$

Jensen's Inequality

Let X be a stochastic variable and let φ be a convex function, then Jensen's inequality states that:

$$E[\varphi(X)] \geq \varphi(E[X])$$

This becomes a strict inequality if φ is strictly convex and $Var[X] > 0$.

Present value calculations are exponential X^t with stochastic variables $X \geq 0$ and exponents $t = 1, 2, 3, \ldots$ This is a convex function, so:

$$E[X^t] \geq E[X]^t$$

Jensen's Inequality, Example

X is either 0.9 or 1.5. Consider X^4

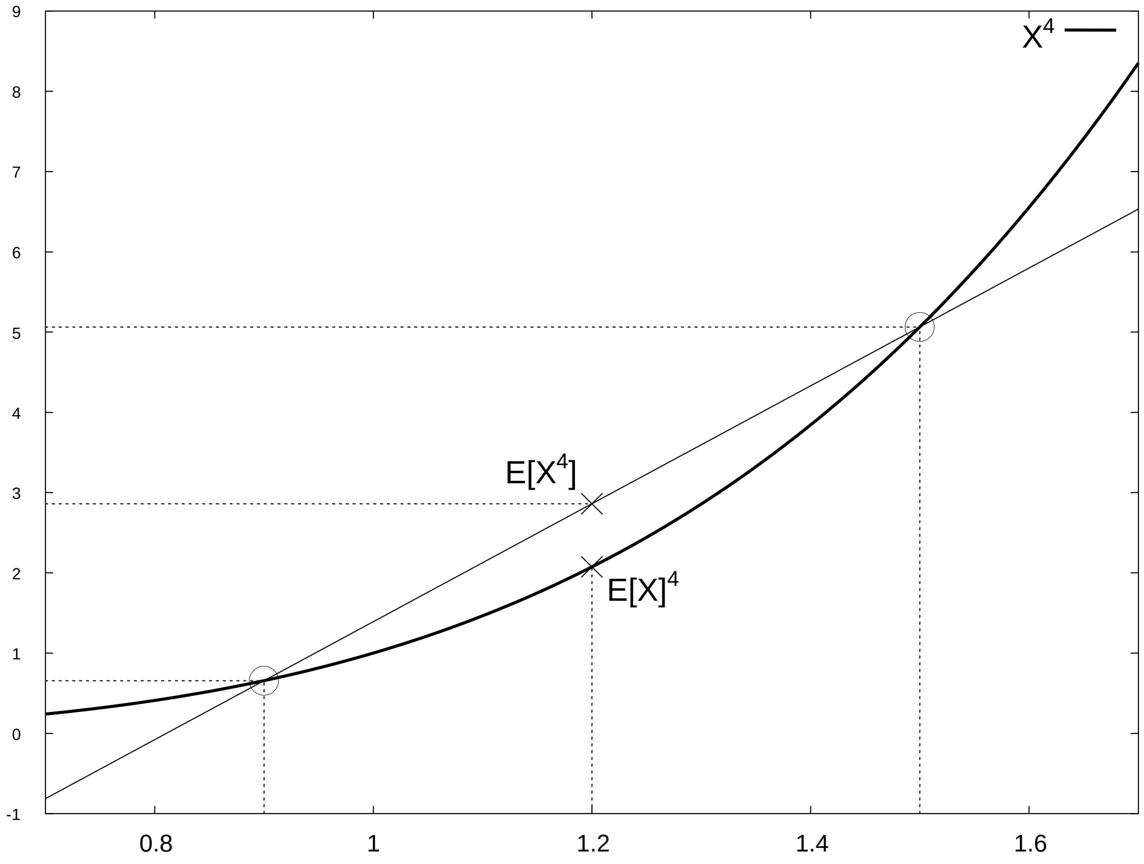
$$E[X^4] = \frac{0.9^4 + 1.5^4}{2}$$

$$= \frac{0.6561 + 5.0625}{2} \simeq 2.86$$

$$E[X]^4 = \left(\frac{0.9 + 1.5}{2}\right)^4 \simeq 2.07$$

So Jensen's inequality is satisfied:

$$E[X^4] \geq E[X]^4$$



Stochastic Discount Rate

Let D be a stochastic discount rate. The stochastic present value is:

$$PV = \sum_{t=1}^{n} \frac{Earnings_t}{(1 + D)^t}$$

This uses an exponential function, so according to Jensen's inequality:

$$E[PV] = E\left[\sum_{t=1}^{n} \frac{Earnings_t}{(1 + D)^t}\right] \geq \sum_{t=1}^{n} \frac{Earnings_t}{(1 + E[D])^t}$$

So using the mean discount rate may underestimate the present value.

Stochastic Growth Rate

Let G be a stochastic growth rate. The stochastic present value is:

$$PV = \sum_{t=1}^{n} Earnings \cdot \left(\frac{1+G}{1+d}\right)^{t}$$

This uses an exponential function, so according to Jensen's inequality:

$$E[PV] = E\left[\sum_{t=1}^{n} Earnings \cdot \left(\frac{1+G}{1+d}\right)^{t}\right] \geq Earnings \cdot \sum_{t=1}^{n} \left(\frac{1+E[G]}{1+d}\right)^{t}$$

So using the mean growth rate may underestimate the present value.

Stochastic Growth & Discount Rate

Let G and D be stochastic growth and discount rates, respectively.

The stochastic present value is:

$$PV = \sum_{t=1}^{n} Earnings \cdot \left(\frac{1+G}{1+D}\right)^t$$

This uses an exponential function, so according to Jensen's inequality:

$$E[PV] = E\left[\sum_{t=1}^{n} Earnings \cdot \left(\frac{1+G}{1+D}\right)^t\right] \geq Earnings \cdot \sum_{t=1}^{n} \left(\frac{1+E[G]}{1+E[D]}\right)^t$$

Stochastic Discount Rate, Example

Let D be a stochastic discount rate that can be either 2%, 10% or 14%.

The average discount rate is $E[D] \simeq 8.7\%$.

Let $Earnings_t = 1$ for convenience.

The mean present value for the first ten years is:

$$E\left[\sum_{t=1}^{10} \frac{1}{(1+D)^t}\right] \simeq 6.78 \geq 6.51 \simeq \sum_{t=1}^{10} \frac{1}{(1+E[D])^t}$$

The actual mean (left) is 4% greater than the estimate (right).

Stochastic Growth Rate, Example

Let G be a stochastic growth rate that can be either -10%, 5% or 20%.

The average growth rate is $E[G] = 5\%$.

Let discount rate be $d = 10\%$.

The mean present value for the first ten years is:

$$E\left[\sum_{t=1}^{10}\left(\frac{1+G}{1+d}\right)^t\right] \simeq 9.45 \geq 7.81 \simeq \sum_{t=1}^{10}\left(\frac{1+E[G]}{1+d}\right)^t$$

The actual mean (left) is 21% greater than the estimate (right).

Stochastic Growth & Discount Rate, Example

Let G be a stochastic growth rate that can be either -10%, 5% or 20%.

Let D be a stochastic discount rate that can be either 2%, 10% or 14%.

Assume they are independent so there are 9 combinations of G and D.

The mean present value for the first ten years is:

$$E\left[\sum_{t=1}^{10}\left(\frac{1+G}{1+D}\right)^t\right] \simeq 10.67 \geq 8.32 \simeq \sum_{t=1}^{10}\left(\frac{1+E[G]}{1+E[D]}\right)^t$$

The actual mean (left) is 28% greater than the estimate (right).

Greater Variance, Example

Let G be a stochastic growth rate that can be either -15%, 5% or 25%.

Let D be a stochastic discount rate that can be either 2%, 10% or 14%.

Assume they are independent so there are 9 combinations of G and D.

The mean present value for the first ten years is:

$$E\left[\sum_{t=1}^{10}\left(\frac{1+G}{1+D}\right)^t\right] \simeq 12.34 \geq 8.32 \simeq \sum_{t=1}^{10}\left(\frac{1+E[G]}{1+E[D]}\right)^t$$

The actual mean (left) is 48% greater than the estimate (right).

Conclusion

- The mean present value may be underestimated when it is calculated from the mean growth and discount rates.
- The magnitude of the error depends on the variance of the stochastic growth and discount rates.

Further Reading

This lecture is based on the paper:

- **Monte Carlo Simulation in Financial Valuation**

Authored by Magnus Erik Hvass Pedersen.

Available on the internet:

www.Hvass-Labs.Org